Lecture 5 - Sep 18

Math Review

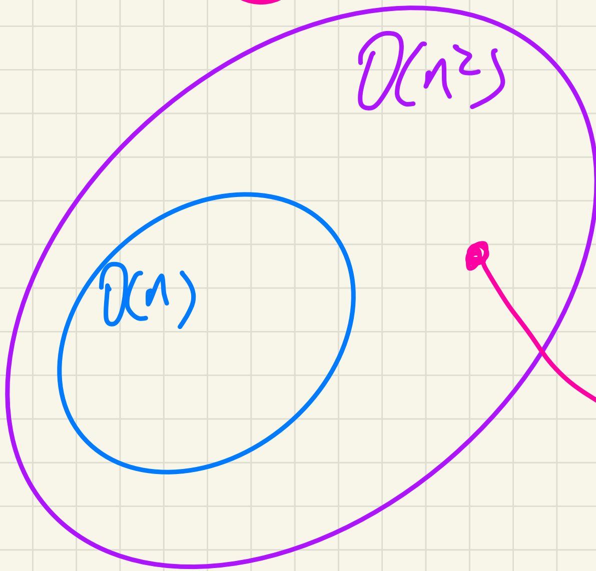
Converting ∀ and ∃ : Equational Proofs
Understanding the Choose Operator
Power Sets

Announcements/Reminders

For next class
precedence of $\wedge$ vs. $\forall$

- Today's class: notes template posted
- Event-B Summary Document
- Priorities:
  - + Lab1 → Due: This Tuesday (Sep 16)
  - + Lab2 → Due: Next Tuesday (Sep 23)
- To be released:
  - + ProgTest guide
  - + 2 Practice Tests
  - + Lab1 solution

$O(n) \subset O(n^2)$



e.g. $2n^2 + 3n - 4$

$n \cdot \log n$

Logical Quantifications: Conversions

$\neg(p \vee q) \equiv \neg p \wedge \neg q$

$R(x)$: $x \in$ 3342_class
$P(x)$: x receives A+

Axiom: $\forall x \bullet Q(x) \iff \neg \exists x \bullet \neg Q(x)$

$$(\forall X \bullet R(X) \Rightarrow P(X)) \Leftrightarrow \neg(\exists X \bullet \overset{R(x)}{R} \wedge \overset{P(x)}{\neg P})$$

$\forall x \cdot R(x) \Rightarrow P(x)$

$\iff \{ \forall x \cdot Q(x) \iff \neg \exists x \cdot \neg Q(x) \}$

$\neg \exists x \cdot \neg (R(x) \Rightarrow P(x))$

$\iff \{ p \Rightarrow q \equiv \neg p \vee q \}$

$\neg \exists x \cdot \boxed{\neg(\neg R(x) \vee P(x))}$

$\iff \{ \neg(p \vee q) \equiv \neg p \wedge \neg q \}$

$\neg \exists x \cdot \neg(\neg R(x)) \wedge \neg P(x)$

$\iff \{ \neg(\neg p) \equiv p \}$

$\neg \exists x \cdot R(x) \wedge \neg P(x)$

$$(\exists X \bullet R \wedge P) \Leftrightarrow \neg(\forall X \bullet R \Rightarrow \neg P)$$

Exercise

✓ De Morgan

Relating Sets: Exercises

$m \in S$ $\quad \neg(m \in S) \iff m \notin S$

$S_1 \subseteq S_2 \wedge S_2 \subseteq S_1 \iff S_1 = S_2$

$\{1, 2\} \subseteq \{1, 2, 3\}$

but they're not equal

$S \subset S$ always fails

↳ not-empty: $\{1, 2\} \subset \{1, 2\}$ ✗

↳ empty: $\emptyset \subset \emptyset$ $\quad \underbrace{|\emptyset|}_{0} < \underbrace{|\emptyset|}_{0}$ ✗

$\emptyset \subset \underline{\underline{S}}$ sometimes holds, sometimes fails

↳ S empty $\longrightarrow$?

↳ S not empty $\longrightarrow$

Sets: Exercises

$S_1 \setminus S_2$ members in S_1 but ~~not~~ in S_2

$\neg(e \in S)$

Set membership: Rewrite $\boxed{e \notin S}$ in terms of $\in$ and $\neg$

Find a common pattern for defining:

1. = (numerical equality) via $\leq$ and $\geq$
2. = (set equality) via $\subseteq$ and $\supseteq$

$x = y \iff x \leq y \land y \leq x$

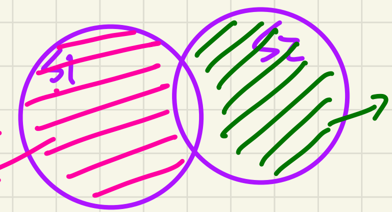
$S_1 = S_2 \iff S_1 \subseteq S_2 \land S_2 \subseteq S_1$

S = {1, 2, 3}, T = {2, 3, 1}, U = {3, 2}

	S		T		U	
S	$\subseteq$	$\subset$	$\subseteq$	$\subset$	$\subseteq$	$\subset$
T	$\subseteq$	$\subset$	$\subseteq$	$\subset$	$\subseteq$	$\subset$
U	$\subseteq$	$\subset$	$\subseteq$	$\subset$	$\subseteq$	$\subset$

Exercise

$S_1 \setminus S_2$



$S_2 \setminus S_1$

$S_1 \setminus S_2 \stackrel{?}{=} S_2 \setminus S_1$

Is set difference $\boxed{(\setminus)}$ commutative?

① Give some witness of violation.

Exercise:

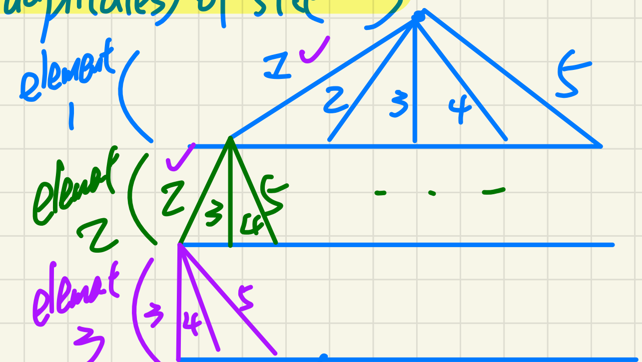
How many **sets** of size 3 can be made out of values 1, 2, 3, 4, 5?

5 $\binom{5}{3}$

(Step 1) Make sequences (with no duplicates) of size 3

$\boxed{5 \times 4 \times 3}$

sequences



(Step 2) Disregard ordering of sequences with the same set of contents

3! For {1, 3, 5}, we would've made sequences:

1 3 5
1 5 3
3 1 5
3 5 1
5 1 3
5 3 1

3 * 2 * 1 = $\boxed{3!}$ sequences that correspond to the same set.

(Step 3) # sets of size 3

$$\frac{\boxed{5 * 4 * 3}}{3!}$$

disregarding ordering.

$\binom{n}{r}$ → n choose r $(n \geq r,\ r \geq 0)$

$\binom{n}{n} = \binom{n}{0} = 1$

out of n given elements, how many ways to make a set of card./size of r?

$$= \frac{n!}{(n-r)! \cdot r!}$$

$$\binom{n}{r} = \binom{n}{n-r}$$

$$= \frac{\underbrace{\boxed{n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot (n-r+1)}}_{}}{r!}$$

r terms

$[n, n-r+1]$

$= r$.

$$\binom{10}{8} = \frac{10 * 9 * 8 * 7 * 6 * 5 * 4 * 3}{8!}$$

$$= \binom{10}{2} = \frac{10 * 9}{2!}$$

Power Set of S (power set)

$$\boxed{\mathbb{P}(S)} = \{ x \mid x \subseteq S \}$$

Each member in $\mathbb{P}(S)$ is a set

Calculate the power set of {1, 2, 3}.

$$\mathbb{P}(\{1, 2, 3\}) = \{x \mid x \subseteq \{1, 2, 3\}\}$$

$$= \left\{ \begin{array}{lll} \emptyset, & \text{smallest} & \binom{3}{0} = 1 \text{ (subset of card. 0)} \\ \{1\}, \{2\}, \{3\}, & & \binom{3}{1} = 3 \text{ (subsets of card 1)} \\ \{2,3\}, \{1,3\}, \{1,2\}, & & \binom{3}{2} = 3 \text{ (subsets of card 2)} \\ \{1,2,3\} & \text{largest} & \binom{3}{3} = 1 \text{ (subset of card 3)} \end{array} \right\}$$

1. smallest member in $\mathbb{P}(S)$: $\emptyset$

$\emptyset \subseteq S$

2. largest member in $\mathbb{P}(S)$: S

$S \subseteq S$

Given a set S, formulate the cardinality of its power set.

$$|\mathbb{P}(\{1, 2, 3\})| = \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3}$$

Set of Tuples

Given n sets $S_1, S_2, \ldots, S_n$, a ***cross/Cartesian product*** of theses sets is a set of n-tuples.

Each ***n-tuple*** $(e_1, e_2, \ldots, e_n)$ contains n elements, each of which a member of the corresponding set.

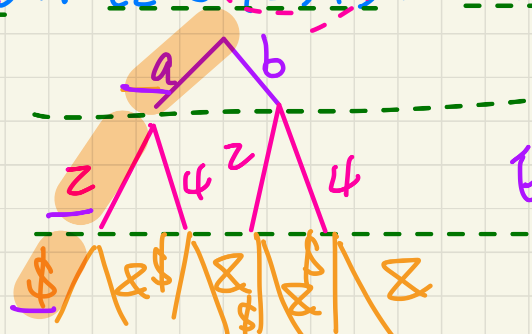
n sets.

$$\boxed{S_1 \times S_2 \times \cdots \times S_n} = \{(\overbrace{e_1, e_2, \ldots, e_n}^{n \text{ elements in the tuple}}) \mid e_i \in S_i \wedge 1 \le i \le n\}$$

Example: Calculate $\{\overset{e1}{a, b}\} \times \{\overset{e2}{2, 4}\} \times \{\overset{e3}{\$, \&}\}$

$= \{ (e1, e2, e3) \mid e1 \in \{a, b\} \wedge e2 \in \{2, 4\} \wedge e3 \in \{\$, \&\} \}$

$= \{ (a, 2, \$) \}$



Each root-to-leaf path corresponds to a ~~n~~ 3-tuple

Relation : set of ordered pairs
tuples.

e.g. a relation on $\boxed{\{1, 2, 3\}}$ and $\boxed{\{a, b\}}$
S T

- Is $(1, a)$ a relation on S and T?

No! $\because$ $(1, a)$ is <u>not</u> a set.